Primes and Primfactors

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ABSTRACT:

The first prime numbers form groups containing all other primes. This is represented first. An analysis of the groups delivers a statement about the distance of prime numbers, an extension of Bertrands postulate, Goldbachs conjecture and an algorithm for computing primes and prime factorization. The results are compared with Riemanns Zeta-Function.

1. Introduction

A number > 1 which has no natural divisors except itself and 1 is called a prime number. There are publicated many treatises about the order of primes. Some of the distribution of primes is shown in the so called Ulam's spiral or spiral of primes[1]. It was found in 1963 by the Polish mathematician Stanislaw Ulam during a scientific lecture scribbling rows of numbers on a paper. He began with the '1' in the center and the he continued in spiral order. When the prime numbers are marked, we find that many primes are on diagonal lines. This is shown in the picture (Fig. 1). If the only even prime number 2 is ignored, the effect is clearer. This picture stimulated me to look into a further systematic arrangement.

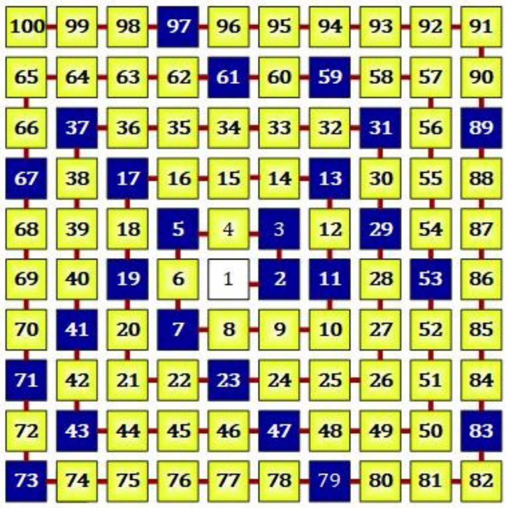

Figure 1: Ulam's Spiral

If the only even prime number 2 is ignored, the effect becomes manifest more obvious. This has me motivate to consider another systematic array.

2. Prime Residue classes

An other systematic arrangement of the primes becomes appearent in certain prime residue classes, also named 'stamps' in the following. Some of its properties will be described now. We denote p as a prime number and p_1 , p_2 , . . . are the primes in ascending order. Analogous to the factorial n! a product of prime numbers (also named primorial) p# is defined by

$$\textbf{Definition (1):} \quad p_n\# = \prod_{i=1}^{n} p_i$$

This is the product of the first n primes. If it doesn't depend on the number n I don't use the index and with $p\#$ I mark such a product.

2.1 The 6-Stamp

It is mentioned in many treatises about primes that all primes > 3 can be written as 6k - 1 or 6k + 1; k = 1, 2, 3, . . This is shown in figure 2 up to the number 30. In the first row, the base line (to which always ranks the '1', are the first 6 numbers. Also each of the following rows contains 6 numbers. The point at the crossing of row and column is the sum of the number at the left of the row and the number on the top of the column. The two circles with white center are the primes 2 and 3. The black points are the other primes < 30. The two numbers 2 and 3 are the prime divisors of 6. The numbers 1 and 5 are denoted as the prime residue classes modulo 6.

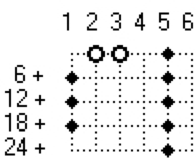

Figure 2: The '6-Stamp' and the primes < 30, ordered in rows of 6 columns

The number of 6 columns results from the product of the first to primes 2 and 3. All prime numbers except 2 and 3 are on the first and fifth row, the '6-stamp'. The numbers on the other columns are divisible either by 2 or by 3.

2.2 The 30-Stamp

In the formation of 30 numbers each, the product of the first 3 primes 2, 3 and 5, the primes (except 2, 3 and 5) are on definite columns. This fact is showed in figure 3 up to the number 210 [[see for example in: "Kurt Diedrich: Primzahl-Lücken-Verteilung ohne Geheimnisse? www.primzahlen.de"]. A grafc representation is chosen to illustrate the correlations and comprehensible for all who are interested on this problem.

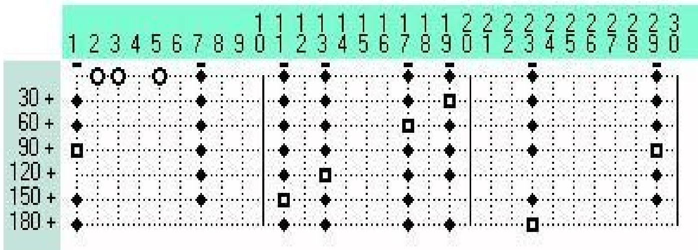

Figure 3: The '30-Stamp' and the primes < 210, ordered in rows 30 columns each

Again the black points are the primes < 210. For reading and destinating a number, on the left side and on the top numbers are marked.(on the numbers at top the numerals are written among themselves). On the crossing of row and column is the number, resulting from the sum of the numbers at the left and the top. The primes 2, 3 and 5, the prime divisors, are pictured as 3 circles with white center. They don't belong to the stamp, the prime residue classes. On the columns of the prime divisors are no other primes. The lines in the figure are dotted. Every tenth line is a continuous line for better orientation. For the numbers with white center I will go in more detail at the end of this section.Wiederum sind die schwarzen Punkte die Primzahlen < 210.
Denoting such a stamp with S, you can describe the 30-stamp with the help of the defnition of $p_n\#$ as $S_{5\#}$. Its elements are the 1 and the primes except 2, 3 and 5 in the frst row, the basic line. In figure 3 they are marked with a dash. They are named $s_1, s_2, \dots ,s_m$. We have 8 columns containing all primes except 2, 3 and 5. Here therefore is m=8. Only the elements in the first row are part of the stamp. All primes > 30 have the form $s_j + 30k$; j = 1, . . . ,m; k = 1, 2, 3, ... In figure 3 there are another 6 lines, which show how to make the next stamp.

2.3 The 2-Stamp

The smallest stamp one can build in this manner consists of two columns. The first row contains the number 1 and the first prime 2. In the second row the next prime 3 is found. Figure 4 shows only 3 rows.

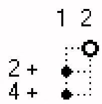

Figure 4: The '2-Stamp' and the primes < 6, ordered in two rows

The first column contains all primes > 2. This fact is known also as the rule: All primes > 2 are odd numbers. Investigating the stamps more exactly you will find that they are built on each other. The numbers of the 30-stamp are generated e.g. by selecting the numbers of the columns 1 and 5 of the 6-stamp in the first 5 rows and eliminating from this selection these numbers, which are divisible by 5. These are the numbers 5 and 25.

2.4 The 210-Stamp

In the same way from the 30-stamp the 210-stamp or $S_{7\#}$ is generated. From the columns of the 30-stamp all numbers up to 210 are selected and those not included in the stamp which without remainder are divisible by 7. So one gets the stamp numbers of the 210-stamp. It can be seen in figure 5 that the prime numbers except 2, 3, 5 and 7 are lying in these columns. By the subset $P_7 = \{2, 3, 5, 7\}$ of the primes this stamp is generated.

Definition(2) Base-primes in the following are named these primes $p_1, p_2, \dots, p_k$ which not belong to the stamp.

For the stamp $S_{7\#}$ these are the primes 2, 3, 5 and 7. Only the '1' and the other marked numbers in the first column belong to the stamp. The primes > 210 up to 2310 are additionaly shown in figure 5.

In order to save space the figure 5 consists of 3 parts 70 numbers each. The first 4 primes are not on the columns of the stamp. The are marked by circles with white center. The numbers marked by squares (white center) are divisible by 11. They are important for the construction of the next stamp. In the stamps $S_{2\#}$, $S_{3\#}$ and $S_{5\#}$ all numbers except 1 are primes. In contrary in this stamp the first time the fact appears that not all numbers of the stamp are prime. This is important for further explanations. They are the numbers 121, 143, 169, 187 and 209 in this stamp . They are products of primes >: 7. All elements of the stamp S7# are indicated with a dash. Accordingly such numbers appear also in higher stamps. In the following I will refer to this stamp $S_{5\#}$ because much of primes can be deduced therefrom.

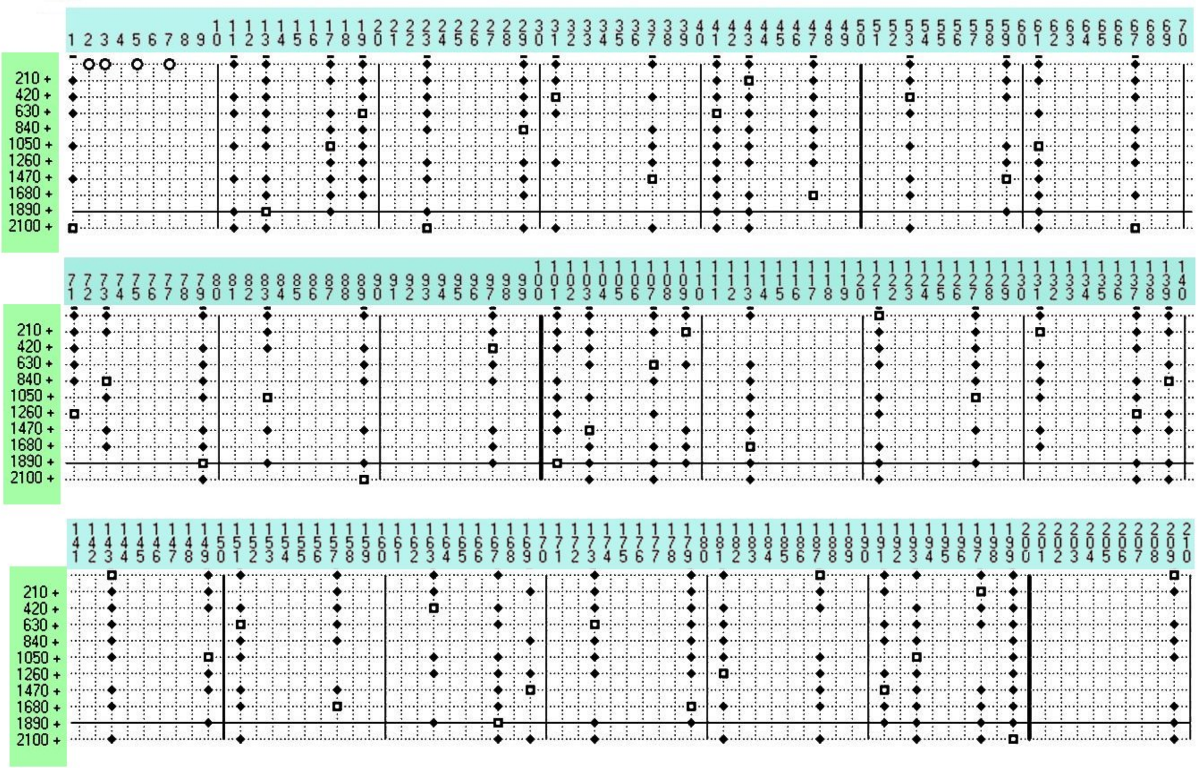

Figure 5: The 210-Stamp and the primes < 2310, ordered in rows 210 colums each.

2.5. The number of elements

First we look at the number of elements in the stamps. The smallest stamp $S_{2\#}$ has only one element $s_1 = 1$.

In the next stamp $S_{3\#}$ there are the primes > 5 on 2 columns. This stamp contains two elements: $s_1 = 1$ and $s_2 = 5$. As $S_{5\#}$ has a formation of 30 we build this stamp from the first 5 rows of S3#. Exactly one of the numbers s, s + 6, s + 12, s + 18, s + 24 on the column s1 is divisible by 5 . The numbers s1; s1 + 6; s1 + 12; s1 + 18; s1 + 24 are also in one column and exactly one of these has the divisor. These two numbers are not included in the stamp $S_{5\#}$. Therefore we get 8 elements in this stamp. If we continue in such procedure and go over to the next higher stamps, we get the following result:

The number of elements of a stamp $S_{p_n\#}$ is $(p_n-1)\# = \prod_{i=1}^{n}(p_i-1)$

As we see on the stamps, for instance on the 210-stamp, the first primes (here 2, 3, 5 und 7) don't belong to the stamp. The stamp however is exclusively formed by these primes. .

Definition(3): The numbers $s_j \leq \prod_{i=1}^{n} p_i$ coprime to p_i; (i = 1, , , n) are denoted as the **Stamp** $S_{p_n\#}$ ist . Ein Stempel $S_{p_n\#}$ wird durch die Primzahlen $p_1, p_2, p_3, , , p_n$ der Teilmenge P_n = 2, 3, , , p_n gebildet. Diese Primzahlen heißen im folgenden die **Primteiler** des Stempels $S_{p_n\#}$.

Definition(4): a mod m is named **prime residue class**, if a and m are coprime, i.e. ged(a,m)=1.

On the stamps it is a matter of prime residue classes. We look for example at the stamp $S_{5\#}$. The prime divisors of 30 are 2, 3 und 5. euler's φ-Function computes the number of positive integers less or equal to $p_{n\#}$ that are coprime to $p_{n\#}p_{n\#}$:

$$\varphi(n) = n * \prod_{p|n}(1 - \frac{1}{p})$$

p|n are all prime divisors of n.

With this formula the number of prime residue classes also can be computed. For example we get

$$\varphi(30) = 30 * [(1 - \frac{1}{2}) * (1 - \frac{1}{3}) * (1 - \frac{1}{5})] = 30 * \frac{4}{15} = 8$$

and the 8 elements are 1, 7, 11, 13, 17, 19, 23 and 29. Exactly these are the elements of $S_{5\#}$ (Fig. 3).

The numbers 1 and $\prod_{i=1}^{n} p_i - 1$ always are part of a stamp. In the stamp $S_{7\#}$ the elements s_j are coprime to the primes 2, 3, 5 und 7. For s_j is valid:

$$s_j \ mod \ p_i \neq 0; \ i = 1, \ . \ . \ . \ , 4; \ 1 \leq s_j \leq 210.$$

The results recur in the next intervals 211 to 420, 421 to 630 etc.. We can therefore speak about these intervals as a copy of the first interval 1 to 210, just like a stamp an impressed pattern copies. Before continuing I will refer to a well known theorem. It is the

Theorem 1: Main theorem of the elementary theory of numbers:

Every number n > 1 , n ∈ N , has exactly one representation

$$n = p_1^{m_1} * \ . \ . \ . \ *p_r^{m_r} = \prod_{\rho=1}^{r} p_\rho^{m_r}$$

with primes $p_1 < p_2 < p_r$ and exponents $m_1 \geq 1$, $m_2 \geq 1$, , , $m_r \geq 1$.
Even Euklid (325-265 BC.) has recognized that a prime number which is a divisor of a*b also is divisor of a or a divisor of b. The main theorem of the elementary theory of numbers has been proved by C. F. Gauss (1777-1855) and E. Zermelo (1871-1953).

We look at a stamp $S_{p_n\#}$ with elements s_j .It has a width of $p_{n\#}$. The first element is s_1=1, but the first prime is $s_2=p_{n+1}$ .If we complete the formation (the first row) to a rectangle with the height p_{n+1} generating the next stamp, in every row in this rectangle exactly one number is divisible by s_2 . The numbers $s_j + k * p_{n\#}$; k=0, 1, 2, 3, . . . , p_{n+1} -1 are on one row of a stamp $S_{pn\#}$. The number $p_{n\#}$ is the product of the primes p_1, , , p_n and p_{n+1} is not a divisor of $p_{n\#}$. All elements s_j are smaller than $p_{n\#}$. By construction of a stamp s_2 is the next prime number, therefore is $s_2 = p_{n+1}$. Either s_j itself has the divisor s_2 . Then the next number divisible by s_2 is $s_j + p_{n+1} * p_{n\#}$ or s_j has not the divisor s_2 . Then s_2 s_j has the divisor s_2 .

Theorem 2: For $p_n\# > 2$ is valid: The stamps are symmetric.

Proof: The oldest method to get the primes is the sieve of Eratosthenes (approx. 300 BC.). The first number only divisible by itself and 1 is the 2, the first prime. First of all all multiples of 2 are eliminated. The next number which is not eliminated is the 3 and therefore it is the next prime number. All multiples of 3 are eliminated etc.. If we apply this method with the primes 2, 3, 5 and 7 up to 210, the product of these 4 primes and also eliminate the prime divisors, we get the stamp $S_{7\#}$. The same result we get by beginning with the four primes on 210 and going backwards and eliminating the multiples of the four prime divisors. Therefore the stamp is symmetric. The same result we get with the first 5 primes 2, . . . , 11 if we pass through the region up to 2310 = 11# forwards and backwards or in general for every product of primes $p_{n\#}$ with n > 1. ❑ (❑ = quod erat demonstrandum)

At 210 the 4 first primes meet together the first time again. In the sieve of Eratosthenes the primes are inserted at 0 and all primes therefore don't touch the number 1. Corresponding to this the pass backwards up to 0 beginning at $p_{n\#}$, n > 2 the number $p_n\# - 1$ is never touched. Therefore 1 and $p_n\# - 1$ always belong to a stamp.

An estimation of the number of primes

A stamp contains all primes until $p_{n\#}$ except the prime divisors. They are smaller than s_2 . In addition a stamp contains the 1, which is not a prime. Therefore is valid:

$$The \ number \ of \ primes \ < p_n\# \ is \ \leq \prod_{i=1}^{n} (p_i - 1) + n - 1$$

Theorem 3: All elements s_j of a stamp $S_{p_n\#}$ form together with the elements $s_j + k * p_n\#$; j = 1, . . . , m ; k = 1, 2, 3, . . . a group th the multiplication as operation.

Proof: The construction of the stamps first eliminates the 2 and all numbers divisible by 2. The next step constructing $S_{3\#}$ eliminates the prime 3 and its multiples etc.. By constructing a stamp $S_{pn\#}$ all prime divisors and his multiples are eliminated. Every number $t + p_n$ in a stamp, which is not an element of the stamp is divisible by one of the prime divisors. Then the numbers $t + k * p_n\#$ are divisible by one of the prime divisors. According to theorem 1 any numbert has exactly one representation in a product of primes. Therefore all elements of a stamp and its products are situated on the columns of the stamp $S_{pn\#}$. ❑

For the smallest stamp $S_{2\#}$ the property of such a group is well known: Products of odd numbers are odd.

Theorem 4: In every stamp S_{pn+1} the numbers $< p_{n+1}{}^2$ are prime numbers.

Proof: The elements of a stamp are representatives of the prime residue classes mod $p_{n\#}$ from the residue-set {0, 1, 2, . . . , $p_n\# - 1$ }, which are coprime to $p_n\#$. In every stamp or this residue-set the first number is the 1, the unity. The second element s_2 is p_{n+1}. The smallest product which we can build in this stamp is $s_2{}^2$ and therefore all integer positive elements $< s_2{}^2$. ❑

3. An Extension of Bertrands Postulate

Bertrand's Postulate is valid for primes:

Theorem 5: (Bertrands Postulate): For every n > 1 , n ∈ N there is at least one prime number between n and 2n .

This theorem has been proved by P. L. Tschebyschow (1821-1894) and later on by J. S. Hadamard (1865-1963) and P. Erdös (1913-1996). .

In the following I want to make a statement about the distance of two prime numbers which is more precise than Bertrand's postulate. For that at first the stamp $S_{7\#}$ is examined. The results are appropriately applicable to all stamps. It is valid:

Theorem 6: In every interval of length $2p_n$ - 2 with upper interval boundary $< {}^{2}{}_{n+1}$ there is at least one prime number.

Proof : I will begin with the stamp $S_{7\#}$ and will show there properties which are also valid for higher stamps. If we compute the modulo-function with the first four primes 2, 3, 5 and 7 in the interval untils 210, we get for the numbers 1 until 210 different results. For the number 63 for example we get the results 63 mod 2 = 1, 63 mod 3 = 0, 63 mod 5 = 3 and 63 mod 7 = 0. I name this result as quadruple {1, 0, 3, 0}. For a number coprime to the first 4 primes, 89 for example we get the quadruple {1, 2, 4, 5}. Because the number 89 is coprime, all 4 numbers are 6 ≠ 0. The stamp $S_{7\#}$ is symmetric in 210, therefore we get for 121, the symmetric number. 1 to 89 the result {1, 1, 1, 2}, i. e. numbers adding with the quadruple of 89 result in the quadruple {2, 3, 5, 7}. These are again the 4 primes. For 147, the number symmetric to 63 we get the result {1, 0, 2, 0}. The corresponding sums are equal to p_i or to mod p_i.

Remark: *The number 121 is not prime but coprime to and therefore an element from $S_{7\#}$.*

		Primzahlen				
		2	3	5	7	
Nr	s_j	s_j mod 2	s_j mod 3	s_j mod 5	s_j mod 7	210 - s_j
1	1	1	1	1	1	209
2	121	1	1	1	2	89
3	31	1	1	1	3	179
4	151	1	1	1	4	59
5	61	1	1	1	5	149
6	181	1	1	1	6	29
7	127	1	1	2	1	83
8	37	1	1	2	2	173
9	157	1	1	2	3	53
10	67	1	1	2	4	143
11	187	1	1	2	5	23
12	97	1	1	2	6	113
13	43	1	1	3	1	167
14	163	1	1	3	2	47
15	73	1	1	3	3	137
16	193	1	1	3	4	17
17	103	1	1	3	5	107
18	13	1	1	3	6	197
19	169	1	1	4	1	41
20	79	1	1	4	2	131
21	199	1	1	4	3	11
22	109	1	1	4	4	101
23	19	1	1	4	5	191
24	139	1	1	4	6	71
25	71	1	2	1	1	139
26	191	1	2	1	2	19
27	101	1	2	1	3	109
28	11	1	2	1	4	199
29	131	1	2	1	5	79
30	41	1	2	1	6	169
31	197	1	2	2	1	13
32	107	1	2	2	2	103
33	17	1	2	2	3	193
34	137	1	2	2	4	73
35	47	1	2	2	5	163
36	167	1	2	2	6	43
37	113	1	2	3	1	97
38	23	1	2	3	2	187
39	143	1	2	3	3	67
40	53	1	2	3	4	157
41	173	1	2	3	5	37
42	83	1	2	3	6	127
43	29	1	2	4	1	181
44	149	1	2	4	2	61
45	59	1	2	4	3	151
46	179	1	2	4	4	31
47	89	1	2	4	5	121
48	209	1	2	4	6	1
		2 – (s_j mod 2)	3 – (s_j mod 3)	5 – (s_j mod 5)	7 – (s_j mod 7)	

Figure 6: Modulo-Values for the elements s_j of the stamp $S_{7\#}$

The elements s_j of $S_{7\#}$ and their quadruples are listed in table 6. In $S_{7\#}$ every possible combination of modulo-values can be found exactly once. This is a consequence of the Chinese remainder theorem. The modulo-values are the remainders to previous multiples of the primes 2, 3, 5 and 7. The functions p_i - (s_j mod p_i)are the differences to the following multiples of p_i. Because of the symmetric position of the elements s_j in $S_{7\#}$ is valid:

$$s_j \ mod \ p_i = p_i - ((210 - s_j) \ mod \ p_i) \ ; \ i = 1, \ . \ . \ . \ , 4; \ s_j \ \in \ S_{7\#}$$

This means: For the numbers 210 - s_j symmetric to s_j we get the differences to the following multiples from the same table (column 210 - sj and lowest r w (grey underlayed). The consequence of this is that the greatest difference of one element of $S_{7\#}$ to a multiple of of p_4 = 7 can be p_4 - 1, i.e. 6. This is valid as well for the difference to the following s_{j+1} as for the differnce to the previous s_j. Therefore is valid: The greatest difference between two elements s_j and s_{j+1} is $2p_4$ - 2 = 12 and therefore in every interval $2ps_4$ - 2 is at least s_j.

Also between every multiple of p_{n+1}, here 11 until the square 121 exist at least one prime number.

In the stamp $S_{7\#}$ between every multiple of 7 exists at least one element s_j. According the chinese residue theorem this is not valid for every stamp $S_{p_n\#}$ or every s_j in the complete intervall of the stamp. Because of theorem 4 we can limit ourselves in $S_{p_n\#}$ to the range ≤ $p^2{}_{n+1}$. Regarding the stamp $S_{2\#}$ we see: All odd numbers up to the square of the next prime 3 are primes. Not till the 9, the square of the next prime 3, this number becomes active, to remove multiples of 3 according to the sieve of Eratosthenes. This is also valid for the next prime number 5 which will be active at 25 (Fig. 2). All products with 5 less 25 are removed by multiples of the primes 2 and 3. Accordant is valid for all other stamps. Therefore the stamp $S_{3\#}$ for example is repeated until $p^2{}_2$ = 25.

If we pass the the next stamp except the number p_{n+1} which becomes prime divisor up to $p_{n+2}{}^2$ exactly 2 numbers are removed from the stamp: $p_{n+1}{}^2$ and p_{n+1} * p_{n+2}. These s_j beeing between $p_{n+1}{}^2$ and p_{n+1} * p_{n+2} must be prime as well as these s_j beeing between p_{n+1} * p_{n+2} and $p_{n+2}{}^2$.

According to these explained properties it is sufficient to proof:

Theorem 7: For n > 2 is valid: Up to n^2 between every multiple of n is at least one prime number.

Proof: I will begin with an example from the prime numbers. I select the stamp $S_{7\#}$ the square of the prime 11 and there the interval]99, . . . ,110[is examined. The boundaries of the interval are to consecutive multiples of 11. This interval should contain as stated at least one prime number. The interval has 6 even and 6 odd numbers.

Remark: *With]a, . . . ,b[an interval is named which contains the numbers greater than a and less then b.The numbers a and b do not belong to this set.*

The even numbers are removed by multiples of 2 and for the odd numbers is valid: Maximal 2 of these can be removed by multiples of 3, maximal one by multiples of 5 and 7 each. This is the worst case that can occur. Therefore from the 5 odd numbers remains at least one. It must be a prime. This is valid for all intervals](i-1) * 11, . . . ,i * 11[for i = 1, . . . ,11. I have chosen here an interval greater 49. For intervals with upper boundary less 49 the number of primes in such an interval increases. Thus the assumption that between to consecutive multiples of 11 up to the square 121 is no prime number leads to a contradiction.

We have here a multiplicative mapping of the odd numbers less than p_n to every interval](i - 1) * p_n, . . . , i * p_n[for i = 2, . . . ,p_n. An odd number which is not prime maps thereby with one of its prime divisors. But there is one exception: The 1, the identity element can't execute a multiplicative mapping. If in the interval only different odd numbers are hit, we have the worst case: One odd number remains which must be a prime number. If in this relation also even numbers are hit or odd numbers multiple are hit, the number of primes in the interval increases correspondingly.

In contrast to addition with the 0 as the neutral element at the multiplication the neutral element 1 belongs to the positive integer numbers and is odd. This has an effect on the upper intervals. The already existing primes are matched to the lower intervals.

This is not only valid for prime numbers, but for all odd numbers > 1. Now to the even numbers: an interval]n, . . . ,2n[with n even contains as much odd numbers as the interval]n+1, . . . ,2(n+1)*n[. An example for intervals with even numbers as interval boundaries if given in figure 7. It is common knowlege, that in every interval]10, . . . , 20[,]20, . . . , 30[until]90, . . . , 100[is at least one prime number. Theorem 7 declares the reason.

Interval Number	Interval	Prime Numbers 3	3*	5	7	worst case	Number of Primes	Prime Numbers in the Interval
1	]10, . . . ,20[	12	15	15	14	+3	4	11, 13, 17, 19
2	]20, . . . ,30[	21	27	25	21	+1	2	23, 29
3	]30, . . . ,40[	33	39	35	35	+1	2	31, 37
4	]40, . . . ,50[	42	45	45	49	+2	3	41, 43, 47
5	]50, . . . ,60[	51	57	55	56	+1	2	53, 59
6	]60, . . . ,70[	63	69	65	63	+1	2	61, 67
7	]70, . . . ,80[	72	75	75	77	+2	3	71, 73, 79
8	]80, . . . ,90[	81	87	85	84	+1	2	83, 89
9	]90, . . . ,100[	93	99	95	91	0	1	97

Remarks:
* Primdivisor of 9
a Mapping on an even Number
b Multiple Mapping

Figure 7: Intervals]10, . . . , 20[until]90, . . . , 100[

On every interval there is a mapping of the odd numbers 3, 5, 7 and 9. The number 1 can't map and the number 9 is mapping by the prime divisor 3. In the ninth interval]90, . . . , 100[the mapping hits on different odd numbers therefore we have there the worst case, i. e there is only one prime number. In the 8. interval the 7 maps on a even number, all other maps are odd and different. There are 2 primes. Another example is the third interval. There the numbers 5 and 7 map on the same number 35 and therefore here are 2 numbers also. All intervals with products $u_j * u_k <$ 10; odd and $u_k < 10$; odd and $u_j * u_k \neq 10$ contains more then one prime number.

At the intervals]11, . . . ,22[etc. the interval wirh the smallest number of primes is the interval]110, . . . ,121[wirh the prime 113. Now back the the higher intervals above mentioned. Here we have many different products of odd numbers u_j and u_k with $u_j < 1000$ and $u_k < 1000$ which are between 1000 and 1 000 000 and are unequal $p_i u^n$ with $p_i < 1000$. Likewise there are some prime numbers less than 1000, for instance those between 800 and 1000. Nearly the half of these prime maps in an interval only once in fact on an even number. A lot of the other half maps in the next interval only on an even number. Therefore the worst case can't occur, i.e every interval contains more than one prime number. The intervals of the size 21, i.e. the intervals]21, . . . , 42[,]42, . . . , 63[until]420, . . . , 441[have already at least 2 prime numbers, in the intervals of the size 100 there are at least 7 and in the intervals of the size 500 there are at least 26 prime numbers.

It is suffcient to examine the intervals up to 1 000 000. This has several reasons:

1. The " 1 " doesn't map.
2. With the function x/ln(x) the number of primes is estimated. >Then the number of primes in the interval]x^2 -x, . . . ,x^2[can estimated with

$$(x^2-1/2)/ln(x^2-1/2)-(x^2-x+1/2)/ln(x^2-x+1/2)$$

. This function is an ascending function (for x > 2), i.e. the number of primes in these intervals is gradually ascending.

Table 8 shows the minimal number of primes in the intervals until]39 800, . . . ,40 000[. In the first row the numbers 0, 1, . . . ,9 are entered, in the first row the numbers 0, 10, . . . ,200.

	0	1	2	3	4	5	6	7	8	9
0+			1	1	1	1	1	1	1	2
10+	1	1	1	1	1	2	2	1	1	1
20+	1	2	2	2	2	2	3	2	3	3
30+	2	2	3	3	3	3	3	2	2	2
40+	2	3	4	3	3	4	3	3	4	3
50+	3	4	4	4	3	3	4	5	4	3
60+	4	5	4	5	4	5	5	5	5	5
70+	5	5	5	5	5	6	7	6	4	5
80+	6	6	5	6	6	6	6	6	7	7
90+	6	6	6	7	7	7	7	6	6	7
100+	7	7	7	8	7	7	8	6	8	8
110+	8	8	8	7	8	8	9	8	9	9
120+	8	9	10	8	9	10	9	10	9	9
130+	10	10	10	10	10	10	10	10	9	9
140+	9	9	10	11	11	10	9	10	10	11
150+	10	12	10	10	11	10	11	11	11	9
160+	10	10	11	11	12	11	12	11	11	12
170+	12	12	13	11	12	12	13	13	12	13
180+	13	11	13	13	12	13	13	13	11	12
190+	12	14	13	14	13	13	12	15	13	12
200+	11									

Figure 8: Minimal number of primes in the intervals until 200 * 200

As an example in the topmost row the number 8 is marked with an grey background. At the numbers in the first column the number 20 has an grey background. At the intersection of the row beginning with 20 an the column beginning with 8 (20+8=28) the number 3 is entered. This number indicates that in every of the intervals]28, . . . , 2×28[,]2×28, . . . ,3×28[until]27×28, . . ,28×28[i.e. between 28 and 56, between 56 and 84 etc. until between 756 and 774 are at least 3 primes. Another example: The number 11 in the last row means: The intervals between 200, 400, 600,. . . , 40 000 contain at least 11 primes.

Looking with a table of prime numbers at higher intervals, for example the intervals]1000, . . . , 2000[bis]999 000, . . . ,1 000 000[, we find in every interval more than 50 prime numbers. The theorem 7 proves theorem 6 also. The theorem is helpful for some problems of prime numbers.

Until p^2 the distance of two adjacent prime numbers is < 2n.
Duality: With theorem 4 and theorem 7 we have dual theorems: At the integers >: 1 of the prime residue group $(Z/pn\#Z)^*$ there are up to p_{n+1}^2 only primes and between every multiple of p_{n+1} there exists up to p_{n+1}^2 at least one prime.

At large numbers it happens that we know a prime number, we call it p_{n+1} for the moment but the previous prime number p_n we don't know. Because $p_{n+1} - p_n \geq 2$ is valid and if we now substitute n for n+1 the following statement is also valid:

Theorem 8: Let be n and n+1 two natural numbers > 1. Then there are between n^2 and $(n+1)^2$ at least 2 prime numbers.

Proof:
1. Between 1 and 4 there are 2 prime numbers : 2 and 3.
2. There exists the formula $(n-1)^2 = n^2 - 2n - 1$. If we interpret these formula for $n \geq 3$ then $(n-1)^2$ is in the penultimate interval (-2n) the first number (+1) relatively to n^2. Therefore the prime number respectively the prime numbers in the penultimate interval must be greater than $(n-1)^2$. Also the last interval before n^2 contains at least one prime number. Consequently we have at least two prime numbers between n^2 and $(n+1)^2$. ❑

4. Goldbachs Conjecture

In 1742 C. Goldbach (1690-1764) remarked in a letter to L. Euler (1707-1783) the conjecture, whether every odd natural number n > 5 can be written as a sum of three prime numbers or whether every even natural number n > 4 can be written as a sum of two prime numbers? Both sentences are known as Goldbach's conjecture.

There are several attempts to prove this conjecture. This is therefore difficult because prime numbers arise from formation of products and the formation of a sum cannot be brought about to connection with that.

At first a graphic representation shall give a general idea. I show the numbers in a straight line in the horizontal and draw in the prime numbers. They are marked as black points in the topmost row.

We form a matrix of the natural numbers of 1 to n . We then mark the prime numbers in a vertical direction in any column which starts with a prime number. We mark the prime numbers also in the first column. We therefore get a symmetric picture of the prime numbers. In figure 9 every fifth number is drawn on the top and on the left side.

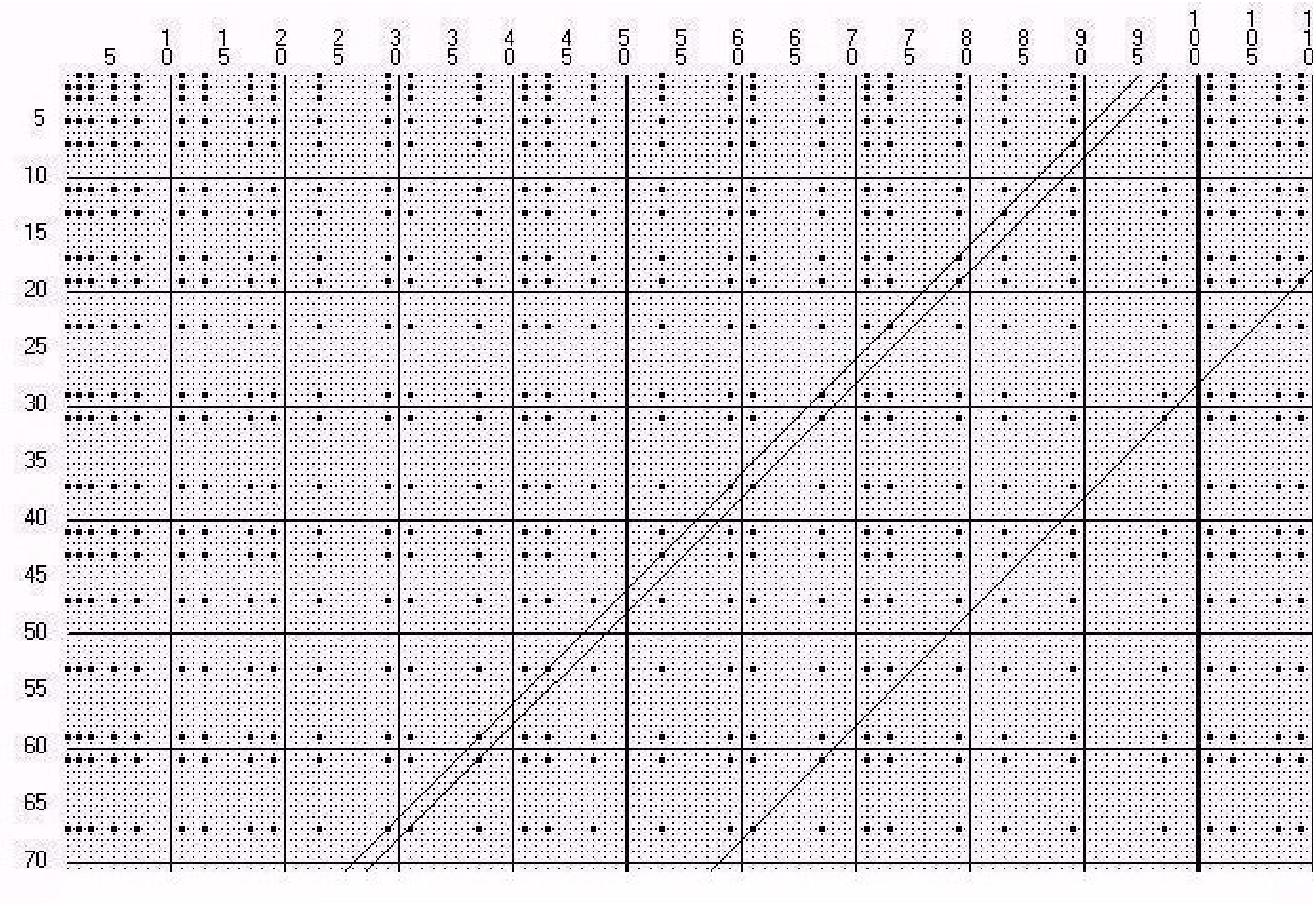

Figure 9: Primes in a two-dimensional array

In the drawing every tenth line is solid for better orientation, the 50th line is a little thicker. A diagonal of 225°(= 45°) beginning at an odd number 2n - 1 in the first row through this numeric array hits prime numbers. One can immediately read solutions for Goldbach's conjecture here. The diagonal which starts at 127 for example hits the number 19 at 109. This two numbers are prime numbers in accordance with the construction and its sum is 128. Going on at the diagonal we get the intersection point 31 at 97, with the sum 128 also and the intersection point 61 at 67.

To find solutions for Goldbach's conjecture, we must start at an odd number with the diagonal because the drawing doesn't contain the zero-row but starts with the number 1. We notice that for diagonals starting at a prime number it is more diffcult to find an intersection point than diagonals starting at an odd number which isn't a prime. This is a way for looking at the problem.

We have shown in the previous chapter that at least 2 prime numbers lie in every interval of the size $2p_{n+1}$. We want to fix one of these prime numbers now. If we do that we still have at least one prime number in the mentioned interval. We want to demonstrate this at the example with the first 4 prime numbers 2, 3, 5 and 7. The filling aerea then has the size 22.

One of the prime numbers is fixed on the 1, i.e. we use the numbers 2, 3, 5 and 7 only in such a way that they don't hit on the 1. Then we can allow the 2 only starting at the 2. The prime number 3 we can also allow to have its starting position on the number 2. It then hits 2, 5, 8 etc. . We however also can put it on 3. It then meets 3, 6, 9 and the other multiples of 3. According to this procedure the prime number 5 can be allowed to start at 2, 3, 4 or 5 and the prime number 7 can be allowed to start at 2, 3, 4, 5, 6 and 7. There is the following rule in the algorithm : If a field in a row is filled with a number > 0 then the field is not changed if a new number will will the same field. This rule is not necessary, it is only for better understanding of figure 10.

row	primes / starting points				Interval from 1 to 22																						zeros	elements of $S_{7\#}$	
	2	3	5	7	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22		fw	bw
1.	2	2	2	2	0	2	0	2	3	2	5	2	7	2	3	2	0	2	0	2	3	2	0	2	0	2	5	209	1
2.	2	2	2	3	0	2	7	2	3	2	5	2	0	2	3	2	0	2	0	2	3	2	0	2	0	2	5	89	121
3.	2	2	2	4	0	2	0	2	3	2	5	2	0	2	3	2	0	2	0	2	3	2	0	2	0	2	6	179	31
4.	2	2	2	5	0	2	0	2	3	2	5	2	0	2	3	2	0	2	0	2	3	2	7	2	0	2	5	59	151
5.	2	2	2	6	0	2	0	2	3	2	5	2	0	2	3	2	7	2	0	2	3	2	0	2	0	2	5	149	61
6.	2	2	2	7	0	2	0	2	3	2	5	2	0	2	3	2	0	2	0	2	3	2	0	2	7	2	5	29	181
7.	2	2	3	2	0	2	5	2	3	2	0	2	7	2	3	2	5	2	0	2	3	2	0	2	0	2	4	83	127
8.	2	2	3	3	0	2	5	2	3	2	0	2	0	2	3	2	5	2	0	2	3	2	0	2	0	2	5	173	37
9.	2	2	3	4	0	2	5	2	3	2	0	2	0	2	3	2	5	2	0	2	3	2	0	2	0	2	5	53	157
10.	2	2	3	5	0	2	5	2	3	2	0	2	0	2	3	2	5	2	0	2	3	2	7	2	0	2	4	143	67
11.	2	2	3	6	0	2	5	2	3	2	0	2	0	2	3	2	5	2	0	2	3	2	0	2	0	2	5	23	187
12.	2	2	3	7	0	2	5	2	3	2	7	2	0	2	3	2	5	2	0	2	3	2	0	2	7	2	3	113	97
13.	2	2	4	2	0	2	0	2	3	2	0	2	5	2	3	2	0	2	0	2	3	2	5	2	0	2	5	167	43
14.	2	2	4	3	0	2	7	2	3	2	0	2	5	2	3	2	0	2	0	2	3	2	5	2	0	2	4	47	163
15.	2	2	4	4	0	2	0	2	3	2	0	2	5	2	3	2	0	2	0	2	3	2	5	2	0	2	5	137	73
16.	2	2	4	5	0	2	0	2	3	2	0	2	5	2	3	2	0	2	0	2	3	2	5	2	0	2	5	17	193
17.	2	2	4	6	0	2	0	2	3	2	0	2	5	2	3	2	7	2	0	2	3	2	5	2	0	2	4	107	103
18.	2	2	4	7	0	2	0	2	3	2	7	2	5	2	3	2	0	2	0	2	3	2	5	2	7	2	3	197	13
19.	2	2	5	2	0	2	0	2	3	2	0	2	7	2	3	2	0	2	5	2	3	2	0	2	0	2	5	41	169
20.	2	2	5	3	0	2	7	2	3	2	0	2	0	2	3	2	0	2	5	2	3	2	0	2	0	2	5	131	79
21.	2	2	5	4	0	2	0	2	3	2	0	2	0	2	3	2	0	2	5	2	3	2	0	2	0	2	6	11	199
22.	2	2	5	5	0	2	0	2	3	2	0	2	0	2	3	2	0	2	5	2	3	2	7	2	0	2	5	101	109
23.	2	2	5	6	0	2	0	2	3	2	0	2	0	2	3	2	7	2	5	2	3	2	0	2	0	2	5	191	19
24.	2	2	5	7	0	2	0	2	3	2	7	2	0	2	3	2	0	2	5	2	3	2	0	2	7	2	4	71	139
25.	2	3	2	2	0	2	3	2	0	2	5	2	3	2	0	2	0	2	3	2	5	2	0	2	3	2	4	139	71
26.	2	3	2	3	0	2	3	2	0	2	5	2	3	2	0	2	0	2	3	2	5	2	0	2	3	2	4	19	191
27.	2	3	2	4	0	2	3	2	0	2	5	2	3	2	7	2	0	2	3	2	5	2	0	2	3	2	3	109	101
28.	2	3	2	5	0	2	3	2	7	2	5	2	3	2	0	2	0	2	3	2	5	2	7	2	3	2	2	199	11
29.	2	3	2	6	0	2	3	2	0	2	5	2	3	2	0	2	7	2	3	2	5	2	0	2	3	2	3	79	131
30.	2	3	2	7	0	2	3	2	0	2	5	2	3	2	0	2	0	2	3	2	5	2	0	2	3	2	4	169	41
31.	2	3	3	2	0	2	3	2	0	2	0	2	3	2	0	2	5	2	3	2	0	2	0	2	3	2	5	13	197
32.	2	3	3	3	0	2	3	2	0	2	0	2	3	2	0	2	5	2	3	2	7	2	0	2	3	2	4	103	107
33.	2	3	3	4	0	2	3	2	0	2	0	2	3	2	7	2	5	2	3	2	0	2	0	2	3	2	4	193	17
34.	2	3	3	5	0	2	3	2	7	2	0	2	3	2	0	2	5	2	3	2	0	2	7	2	3	2	3	73	137
35.	2	3	3	6	0	2	3	2	0	2	0	2	3	2	0	2	5	2	3	2	0	2	0	2	3	2	5	163	47
36.	2	3	3	7	0	2	3	2	0	2	7	2	3	2	0	2	5	2	3	2	0	2	0	2	3	2	4	43	167
37.	2	3	4	2	0	2	3	2	0	2	0	2	3	2	0	2	0	2	3	2	0	2	5	2	3	2	5	97	113
38.	2	3	4	3	0	2	3	2	0	2	0	2	3	2	0	2	0	2	3	2	7	2	5	2	3	2	4	187	23
39.	2	3	4	4	0	2	3	2	0	2	0	2	3	2	7	2	0	2	3	2	0	2	5	2	3	2	4	67	143
40.	2	3	4	5	0	2	3	2	7	2	0	2	3	2	0	2	0	2	3	2	0	2	5	2	3	2	4	157	53
41.	2	3	4	6	0	2	3	2	0	2	0	2	3	2	0	2	7	2	3	2	0	2	5	2	3	2	4	37	173
42.	2	3	4	7	0	2	3	2	0	2	7	2	3	2	0	2	0	2	3	2	0	2	5	2	3	2	4	127	83
43.	2	3	5	2	0	2	3	2	5	2	0	2	3	2	0	2	0	2	3	2	0	2	0	2	3	2	5	181	29
44.	2	3	5	3	0	2	3	2	5	2	0	2	3	2	0	2	0	2	3	2	7	2	0	2	3	2	4	61	149
45.	2	3	5	4	0	2	3	2	5	2	0	2	3	2	7	2	0	2	3	2	0	2	0	2	3	2	4	151	59
46.	2	3	5	5	0	2	3	2	5	2	0	2	3	2	0	2	0	2	3	2	0	2	7	2	3	2	4	31	179
47.	2	3	5	6	0	2	3	2	5	2	0	2	3	2	0	2	7	2	3	2	0	2	0	2	3	2	4	121	89
48.	2	3	5	7	0	2	3	2	5	2	7	2	3	2	0	2	0	2	3	2	0	2	0	2	3	2	4	1	209

Figure 10: The starting positions of P7 = {2, 3, 5, 7} not striking the number 1

Altogether 48 filling areas arise in this way. This is represented in table 10. The first row in the range of 1 to 22 of the table is made as follows: At first the range of 1 to 22 is made empty, i.e. filled with zeros. The prime number 2 is used on 2. The number 2 and its multiples fill the numbers 2, 4, 6, . . . , 22. The prime number 3 is also used on 2. It and its multiples fill then the numbers 2, 5, 8, 11, 14, 17 and 20. The prime number 5 is also used on 2, just as the prime number 7. The fist row is now completed. In the second row, with the prime numbers 2, 5 and 7, we proceed just as in the 1st row. Merely the prime number 3 starts on position 3. It and its multiples then occupy the fields 3, 6, 9, 12, . . . ,21. In a corresponding way the additional rows are made. The last row is remarkable. The prime numbers are used exactly just like we know it from the sieve of the Eratosthenes. As we see there are 48 possibilities and therefore we can have a try to assign these to the elements s_j of $S_{7\#}$. At first to the last row. It can undoubtedly be assigned to the element 1 of $S_{7\#}$. The elements in $S_{7\#}$ aren't divisible by the first 4 prime numbers as we know. Furthermore every element has the following property: It has a definite difference to the next multiple of the prime numbers 2, 3, 5 and 7. I would like to explain this in two examples:

Example 1: For the element 83 the next multiples of 2, 3, 5 and 7 lie at 84, 84, 85 and 84. so the differences to 83 are 1, 1, 2 and 1. These correspond to the starting points 2, 2, 3 and 2 in figure 8. This is the seventh row. Therefore the 83 is there registered under *fw* (=forward).

Example 2: For the element 43 the next multiples of 2, 3, 5 and 7 are 44, 45, 45 and 49. The differences to 43 therefore are 1, 2, 2 and 6, i.e. they correspond the starting points of 2, 3, 3 and 7. This is the 36th row. Hence the number 43 is registered under *fw*.

In this way the entries in column *fw* (=forward) can be found for sll elements of $S_{7\#}$. In the same way the differences to the next lower multiples of $P_7 = \{2, 3, 5, 7\}$ in $S_{7\#}$ can be determined. They are listed in the column *bw* (=backward). Because of the symmetry of $S_{7\#}$ they also can be computed as $bw = 7\# - fw$. Since the assignment of a row of figure 10 to an element of $S_{7\#}$ is unique, we can draw the conclusion also in the opposite direction: To every element bw from $S_{7\#}$ only one row of the table 10 can be assigned. There is at least one prime number in the interval from 1 to 22 in every row of table 10. The number *bw* assigned to this row hits an element s_j of $S_{7\#}$. It holds:

$$p_k + s_j = bw + 1 \qquad (1)$$

In an example we compare the results of figure 10 with figure 9. One of the diagonals marked starts at 97. A comparison with the results of table 10, row 12 ($bw = 97$) shows: The first zero in the interval 2 to 22 is on position 9 (no prime number) corresponds to the crossing of the diagonal at 89. The next zero on position 15 (no prime number) corresponds to the crossing of the diagonal at 83. The next zero on position 19 (prime number) corresponds to the hitting of the diagonal at 79.

Beginning with the subset $P_7 = \{2, 3, 5, 7\}$ of primes in the corresponding stamp/prime residue classes all numbers less then 121, the square of the next prime number are prime. Corresponing statement ist valid for the numbers of figure 10. Here only the values greater than 49 are of interest. In summary, the statement: in the filling aerea 2, . . . ,$2p_{i+1}$ is at least one prime number, we can apply only for the primes between 49 and 121.

Figure 10 is an example for the prime numbers $P_7 = \{2, 3, 5, 7\}$ of the stamp $S_{7\#}$. The construction of such a table can also be carried out for every higher stamp. In accordance with theorem 4, s_j is a prime number if $s_j < s^2$. By a suitable choice 3 of a higher stamp we can achieve that s_j is a prime number in (1).

Remark: *If we look at larger prime number $p_i = 16000057$ for example, this is contained in the stamp $S_{23\#}$ $23\# = 22309870$ is greater than p_ii . The next prime number to 23 is 29. This however doesn't mean that the next even number 16 000 058 has to be a sum of 2 primes one of them is smaller than 58(= $2p_k$). Only if we use a stamp $S_{3989\#}$, all numbers in the stamp which are smaller than 16 000 057 are prime numbers. Then the second element of the stamp is $s_2 = 4001$ and all stamp elements < 40012 = 16008001 are prime numbers. Then it must be a solution for the Goldbach's conjecture one of the two prime numbers is smaller than 8002.*

We have seen now that an element of $S_{7\#}$ corresponds to a variation of the **base prime numbers** $p_i = 2, 3, 5, 7$; p_i, (i = 1, . . . n); n = 4 under the condition that a prime number is fixated. At least another prime number lies in the range of 2 to 22 (= $2p_{i+1}$) if the other is fixated. From this follows that a solution of Goldbach's conjecture for the corresponding element which is listed in the *bw* also exists and to be more precise in the range of $2p_{i+1}$, the filling aerea.

Further solutions of the Goldbach problem are possible, at which the lower of the two prime numbers is greater than $2p_{i+1}$. There is rather at least one solution, wherein the lower of the two numbers is less then $2p_{i+1}$.
For higher numbers a corresponding table like table 10 can be constructed and determined the values *fw* and *bw* in the same way. An element s_j of the stamp and one in the interval $2p_{i+1}$ existing prime number form the sought-after sum, if in the stamp only numbers < p_{i+1}^2 are elected.

By fixation on the 1 in figure 10 it becomes appearent (similiar as in theorem 7): We have here a multiplicative mapping of the primes up to 7 on the stamp $S_{7\#}$. The mapping of the 1 is by the construction explicitly excluded (except in the last column), i. e. the products 1*3, 2*3, 3*3, etc., as well as 1*5, 2*5, 3*5 and 1*7, 2*7, 3*7 are excluded. From that follows that in every of the 48 combinations not only products of the numbers 3, 5 and 7 can be found, but at least one number is not such a product and therefore a prime number.

By a suitable choice of the stamp it can be reached that $s_j < s_j^2$ and therefore a prime number. For even numbers 2n, when 2n - 1 is a prime, the assertion is proved.

The second case: 2n - 1 is not a prime number. We look at a stamp $S_{p_n\#}$ again and in it at the prime numbers < $p_{n+1}^2 = s_2$. Let p_j such a prime number. It was demonstrated in case 1 that p_{j+1} is the sum of 2 prime numbers. By addition of p_j and the primes < $2p_{n+1}$ we get the even numbers until to the next prime number. In the interval $[1, \ldots, 2p^{n+1}]$ however not all odd numbers are prime. There also here are products of the base-primes. These are the numbers 9, 15, 21, ... As we can fixate one prime number on the 1 we can fix a prime number on any odd number in the interval $2p_{i+1}$ and at least one number in this interval remains as prime. The fixation on the prime numbers 3, 5, 7, etc. is unnecessary because these numbers together with p_j fullfills Goldbach's conjecture. If we carry out the fixation on one of the numbers 9, 15, . . . , we get a prime number < $2p_{n+1}$ which together with a prime number < p_j fullfils Goldbach's conjecture. This completes the proof of Goldbach's conjecture.

The proof of the second part can be simplified : The results before have shown that the solution of goldbachs conjecture deals with the residues r_j of a prime p_j to the base-primes $p_1, p_2, p_3, \ldots, p_k$. The the residues of the number p_j - 2 relative to the base-primes are the numbers r_j - 2 or $p_j - r_j - 2$, if $r_j < 2$. For p_j - 2 one could carry out a fixation. Asuming that p_j - 2 is not a prime the diagonal in figure 9 beginning at p_j must be rised at 2. Then the diagonal for p_j - 2 begins at -1 with the same sequence of base-primes and zeroes as computed at p_j. Then the resulting interval for p_j-2 is

[-1, . . . ,2p_{n+1} - 2] instead of [1, . . . ,2p_{n+1}]. This subtraction of 2 is repeated until the previous prime p_{j-1} is reached, i.e. the first zero in the sequence is reached. Then the action described in the first part of the proof begins again. This is demonstrated for the numbers 98, 96, 94 and 92 in figure 11.

	Primzahlen							P	P		P		P				P		P				P		P			
N	Intervall (1-22)						1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
97	.	.	.	.	.	.	0	2	5	2	3	2	7	2	0	2	3	2	5	2	0	2	3	2	0	2	7	2
95	.	.	.	.	0	2	5	2	3	2	7	2	0	2	3	2	5	2	0	2	3	2	0	2	7	2		
93	.	.	0	2	5	2	3	2	7	2	0	2	3	2	5	2	0	2	3	2	0	2	7	2				
91	0	2	5	2	3	2	7	2	0	2	3	2	5	2	0	2	3	2	0	2	7	2						

Figure 11: Computing Goldbachs Numbers for 98 - 92

I will demonstrate this fixiation on an example. We look at the number 147. It has the prime representation 3 * 7 * 7 . The greatest prime number < 147 is 139. Drawing a diagonal with 45° beginning at 147 (Fig. 9) intersects the number 139 at 9. The number 147 is in the interval in which all primes are generated by the base-primes 2, 3, 5, 7 and 11. We will therefore vary these 5 base-primes in the interval [1, . . . ,26] , that they not hit the number 9: The number 2 always must start at 2. The number 3 must start at 1 or 2 but not on 3 because then it would hit the 9. The number 5 can start at 1, 2, 3 and 5. Accordingly the number 7 on 1, 3, 4, 5, 6 and 7 and the number 11 on 1, 2, 3, . . . , 8, 10 and 11. Totaly the are 480 variations. Not all of these variations are listed in figure 12 but only the one of interest.

primes					Interval from 1 to 26																									
2	3	5	7	11	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
starting points																														
2	1	3	1	5	3	2	5	2	11	2	3	2	0	2	0	2	3	2	7	2	0	2	3	2	0	2	5	2	3	2

Figure 12: Fixation of the subset P_7 = {2,3,5,7} for the number 147

The differences of 147 to the preceding multipes of 2, 3, 5, 7 and 11 are 1, 0, 2, 0 and 4. This corresponds to the selected positions of action in figure 9. The primes 11 and 17 are not hit. They correspond to the primes 137 and 131 if we follow further the diagonal beginning at 147.

It isn't necessary to build the table of all possibilities to find the sought-afterprime number for Goldbach's conjecture. For the algorithm it rather suffices to find the prime factors of *2n - 1* and the idfferences of 2n -□ 1 to the preceding multiples of the base primes numbers which are not prime factors of *2n - 1* i.e. the remainders of *2n - 1* with regard to the base-primes. With these numbers we can determine the starting positions for a computation like that in figure 10. The prime numbers p_a there which are marked with a zero are together with *2n -*□ $_a$ solutions of Goldbach's conjecture for *2n*.

Goldbach's numbers must so to speak pass 2 sieves behind each other. The first sieve is the sieve of Eratosthenes at which all prime numbers start at zero. The second sieve is built up similarly as the first one but the base prime numbers don't start its run all at zero but at other starting positions, which can be computed with the remainders of *2n - 1* .

Sierpinksi mentions in his book about the theory of the numbers [[Waclaw ierpinski: Grundlegende Theorie der Zahlen, Wasawa 1964] in the section over Goldbach's conjecture the problem of the differences of prime numbers. Using the above explanations the theorem, that every even number is the difference of two prime numbers also can be proved with the help of the numbers *fw* and the diagonals in the direction of the 315° (not marked in fig. 9).

5. An algorithm for computing prime numbers

In the previous section the elements of S7# and in figure 6 the residues res_i are listed. With figure 12, the table with the fixation of one prime number to position 1, the starting positions (they are named now pos_i) of the 4 prime residues 2, 3, 5 and 7 are showed. The number of these prime divisors i.e. "base-prime numbers" is n=4 for the moment. They lead in every line to a fixation on the number 1. A comparison of the both tables shows, that the values of res_i and pos_i differ by 1. There is $pos_i = res_i + 1$.

The zeros in the filling area i.e. the zeros in a line of figure 12 indicate, where - starting on an element s_k of $S_{7\#}$ a previous element of $S_{7\#}$ is situated. For the element s_k, which are less than 121, the square of the next prime number this means that we in $S_{7\#}$ with this method one can find prime elements. this is valid at least until 49, the square of prime number 7.

Conversely because of symmetry of the stamps/prime residue classes the prime numbers following the element s_k can be found. For that the starting positions in a line of table ?? must be differently elected. They must by be especially complementary to the described values. We must elect as starting positions the values $pos_i = p_i - \square res_i + 1$ and apply the algorithm described in the previous section.

An array-element of the filling aerea can multiple occupied by prime residues. In the previous section onlya single occupation of a field-element was admitted. This was only practised to clarify the algorithm. To accelerate the algorithm one can leave out this examination and in addition restrict to the array-elements with odd array-index only.

The number of array-elements of a filling aerea is for the moment $_{i+1}$. If the filling aerea is mapping an area, in which the square p_{n+1}^2 is situated, the corresponing element in the filling aerea is zero also. The element corresponds not with a prime and must be ejected.

The last zero one can find in the filing-aerea 1, . . . ,$2p_{i+1}$ marks the prime number p_{neu}, if it will not correspond the prime-square p^2_{n+1}. With this new prime number we can continue in the next step. We compute the difference to the previous starting value and add the difference to every of the residues res_i.

After crossing p^2_{n+1} (here=121) we are going to the next stamp, i. e. in the example to $S_{11\#}$ and prime number 11 is admitted to the "base-prime numbers". The the number of "base-prime numbers" is n=5. The residue res_5 for 11 at 121 is zero. The the number of the array elements of the filling aerea becomes 26. The residue res5 for 11 at 121 is zero.

This however is the only number, one must pay attention during the election. Alltogether we need at the computing of prime numbers with this method only a few dividing operations. We have to fill a $2p_{i+1}$-area (Filling-Area) with the base primes p_i and the prime numbers are falling out like the balls from a gumball machine. Only on one number we must have attention, p^2_{n+1}. It is not a prime number but this number increases, if crossed, the number of base primes by 1 and increases the filling aerea.

I will demonstrate the algorithm by an example: I begin with the prime number 97. The prime numbers < √97 are 2, 3, 5 and 7. They are in the figures 11 and 12 the base prime-numbers. The next prime number is 11. The number of the elements of the filling aerea is 2 * 11 = 22. The residues of 97 relatively to 2, 3, 5 and 7 are 1, 1, 2 and 6. We compute the complements. They are 1, 2, 3 and 1. (The algorithm can begin at any number $\geq p_n^2$ (here $\geq$ = 49), if there the residues or complements to the primes $p_1, \ldots, p_n$ are known and the next boundary p_{n+1}^2 (here = 121) is respected.) Alltogether summarized for starting value 97 in table 13.

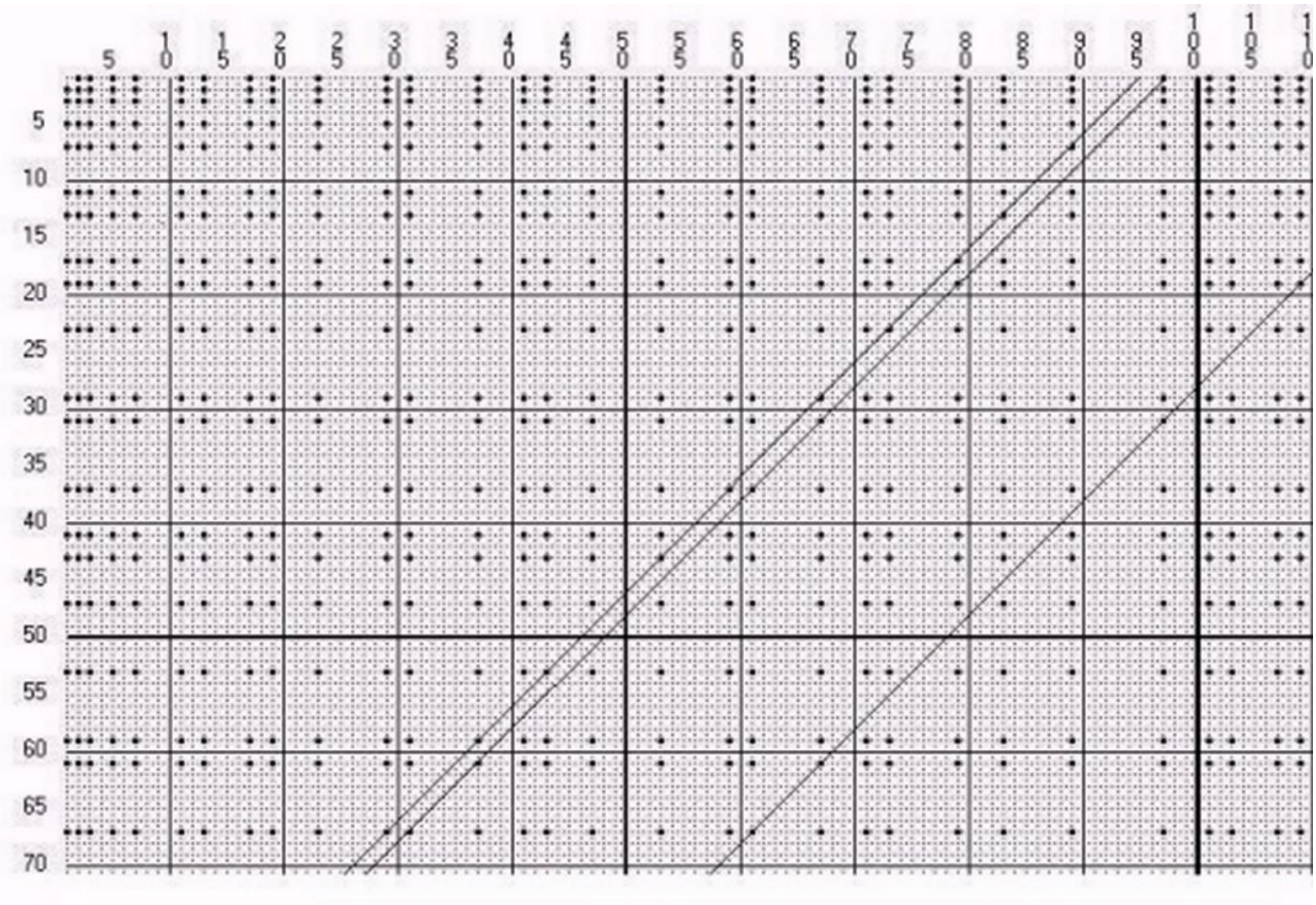

Figure 13: Starting number 97. Primes and modulo-values

The values "Complement" + 1 are the starting positions for the algorithm described in the previous section. In table 13 one can see the result. Here is - except the first value - the 5., 7., 11., 13. and 17. value zero. That means: 101, 103, 107, 109 and 113 are prime numbers.

The last prime number of the series is 113. It is the starting value for the next computing step. By computing with the modulo-function or by subtraction at the end of the filling-aerea the residues and their complements can be computed at the last prime of the filling aerea. These complements are the new starting positions, etc..

```
==========================================================================================
      4 Primes     |        22 Elements of Filling Aerea
-------------------+----------------------------------------------------------------------
     2  3  5  7    |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22
-------------------+----------------------------------------------------------------------
 Starting-Positions |     Primes on the Elements of the Filling Aerea
-------------------+----------------------------------------------------------------------
     2  3  4  2    |  0  7  3  5  0  3  0  2  7  2  0  3  0  5  3  7  0  3  5  2  3  2
==========================================================================================
```

Figure 14: Starting Number 97. Primes and filling aerea.

With 113 as starting value we get the resulting primes 127 and 131. Also the 9. Value, which corresponds to the number 121 contains a zero. This we had exspected and this value is ejected (i.e. is not prime) The next such value would be 143, the product 11 * 13. This however is out ot the filling aerea. Two primes 127 and 131 are crossing the prime-square 121 and the next computing step with the 131 as the starting value contains the 5 base primes 2, 3, 5, 7 and 11 and the numbers of array-elements is 26. The residue of 131 relatively to 11 is 10. This is computed as the difference to the prime-square 121.

The algorithm requires 20 steps from 97 to 811 and there are 116 prime numbers, 40 steps from 97 to 2221 and there are 311 prime numbers.

One can improve the algorithm: If the difference p_{n+1} - □p_n is greater 2pn, the filling aerea can be increased. That's also possible to organize the filling aerea in the algorithm that in one computing step all primes between p_n^2 and p_{n+1}^2 are found.

6. An algorithm for prime factorization

In the preceding section is shown how from a filling aera primes can be taken. On the coresponding positions zeros are found. The other positions in the filling aerea contain primes. These entries are prime factors of the corresponding number. Therefore we must choose the filling aerea that the number z whose prime factors we want to compute is situated between two primes-squares following each other. From the previous section is evident that not only primes are indicated by a zero but also squares of primes. We are here for example in the group of prime residue classes $(Z/210Z)^*$ or in other words in the stamp $S_{7\#}$. There $p_n^2 = 121$ and $p_{n+1}^2 = 169$ are elements also. Therefore we must choose p_n^2 and p_{n+1}^2 that $p_n^2 < z < p_{n+1}^2$. The corresponding filling aerea can begin at p_n^2 and end at p_{n+1}^2 or earliest end at z. The true filling aerea begins always at 1 as shown above. After filling the entries at z - respective the position in the filling aerea which corresponds to z - are prime factors of z.

One has to continue from prime-square to the next prime-square until the condition mentioned above is fulfilled. At the end of every filling aerea, i.e. the interval [$p_n^2, \ldots, p_{n+1}^2$], the residues and complements for the next computing step are determined. The base-prime numbers are increased by one prime number.

Of course we get not all prime factors. Multiple entries of a prime factor can not be entries at the element corresponding z. Also prime factors > are not entered.

It would be an immense computational effort to compute all primes beginning at the first. Thefore is recommended at certain distances to store the complements. For example at the j-th prime p_j^2 to store all complements, which are necessary to continue the computing at p_j^2. .

Imagine the primes orderd in horizontal lines: On the first line the prime 2 lies located and is repeated at 4, 6, 8, ... The prime number 3 is located in the next line. It is repeated at 3, 6, 9, ... In corresponing with the following primes the same is done. On the i-th line the prime pi is situated and is repeated at $2p_{>i}$, $3p_{>i}$, ... Every prime $p_{>i}$ brings an influence to the forming of primes starting at p_i^2 . With the forth prime for example we find that the numbers 49, 77 and 91 are not belong to the primes.

The presented algoritm takes a vertical cut across these lines. This cut reaches always until the i-th line. For the numbers ≥ 49 until numbers < 121 for example this cut is performed until the forth line. Hits this cut not an entry then the number is a prime. Hits the cut however one or more numbers, the these numbers are prime divisors. At every cut a prime divisor occurs only once.

In the algorithm on can add several corrections to accelerate the computing procedure:
Going without the prime factors of the even numbers one can leave out the even numbers in the filling aerea. Similiar the numbers which have the the prime factors 3, 5, 7 and others. I refer hereto on the criteria of divisibility in [Prof. Dr. Harald Scheid: Zahlentheorie, ISBN 3-411-14841-19], p. 121 ff.

Because at the filling every second number is even, on can hurry up the algorithm leaving out at these positions.

If the primes and complements at p_i^2 are computed and stored, sometimes it make sense to pass the computing-process backwards, if e.g. the prime factors for a number between p_{i-1}^2 and p_i^2 should be found.

Further activities are possible to accelerate the algorithm. If the filling aerea is too large and can't handled as a whole in the computing program, it can be divided in severals parts. If the primes > p_{i+1} are not of interest, the procedure can be continued, until the primes p_{i+1} are found. Then p_i^2 greater than p_{i+1} is reached and the residues and complements there are known. Now the residues at p_i^2 relating the base-primes can be computed in another way. Then the prime factors in the filling aerea $p_i^2, \ldots, p_{i+1}^2$ are computed in the manner described above.

In an example the prime divisor of 10177 is calculated. There are a lot of calculations are stored in a pdf-file. A link therefor can be found at the bottom of this article.

7. Comparison with the Zeta-Function

Comparing these results with Riemanns Zeta-Function the following is to note: The computation of the primes occurs between p_i^2 and p_{i+1}^2, by using the previous results - the primes until p_i and the residues resp. complements to the primes at p_i^2 are used to determine the primes until p_{i+1}^2. With the algorithm described in the last chapter on p^2_{i+1} meets no number i.e. it is initially a prime number. Only after the filling the field between p_i^2 and p_{i+1}^2 the prime p_{i+1} is inscribed. At the Zeta-Function operating with the reziproke values of the primes, the real part of the zeros have the value ½. This corresponds here in the described algorithm to a stop at the value p_{i+1}^2. There the first time the prime p_{i+1} comes in. Between the squares of consecutive primes, even between the squares of consecutive positive integers primes resides as shown with theorem 2. The computation of primes stops always only at p_{i+1}^2 so as the Zeta-Function at a zero with the real part ½. Therefore is the conjecture that the zeros of the Zeta-function coincide with the prime squares.

8. End

I would like to point out to Ulam's spiral here again mentioned already at the beginning. The squares of the odd numbers beginning at 1 lie in the diagonal to the top right. In the diagonal beginning at 4 to down on the left are the squares of the even numbers. From theorem 8 results that at a half turn of the spiral we reach one of these diagonals passing at least two prime numbers. A line vertical to this diagonale through the numbers 6, 20, 42, . . . , i.e. $n^2 - n$, (n odd), respectively on the other side though through the numbers 12, 30, 56, . . . , i.e. $n^2 - n$, (n even) marks the boundaries of the intervals. Between these boundaries and the squares there is at least one prime number.

On these points (one number later) the direction of the Ulam-spiral is chanced by +90° rsp. by - 90° depending on the spiral is drawn clockwise or counterclockwise.It is irrelevant whether the spiral begins with the 2 on the right, the left, above or below the 1. If the Ulam-spiral begins with zero in the center, then it changes the direction by 90° exactly at the edges.
If you agree my proves or you have found errors in it please send me an email.